Ch. 11: Rotational Vectors & Angular Momentum
Tuesday March 17th

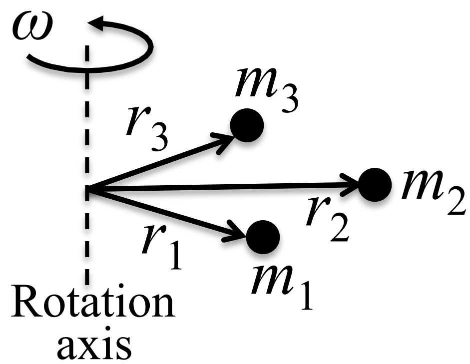
- Quick review of rotational inertia and kinetic energy
- Torque and Newton's 2nd law
- Angular momentum
- Conservation of angular momentum
- Summary of translational/rotational equations
- Examples, demonstrations and *i*clicker
- Return mid-term exams

- Normal schedule for the next several weeks.
- Material covered today relevant to LONCAPA due tomorrow.
- Next Mini-Exam on Thursday 26th (LONCAPA #13-17).

Reading: up to page 182 in Ch. 11

Kinetic energy of rotation

Consider a (rigid) system of rotating masses (same ω):



$$K = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \tfrac{1}{2}m_3v_3^2 + \cdots\cdots$$

$$= \sum \tfrac{1}{2}m_iv_i^2$$

where m_i is the mass of the ith particle and v_i is its speed.

Re-writing this:

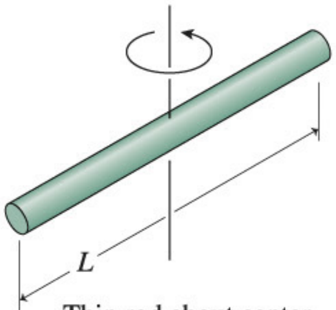
$$K = \sum \tfrac{1}{2}m_i(\omega r_i)^2 = \tfrac{1}{2}\left(\sum m_ir_i^2\right)\omega^2$$

The quantity in parentheses tells us how mass is distributed about the axis of rotation. We call this quantity the rotational inertia (or moment of inertia) I of the body with respect to the axis of rotation.

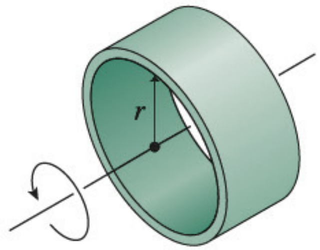
$$I = \sum m_ir_i^2 \qquad\qquad K = \tfrac{1}{2}I\omega^2$$

Rotational Inertia for Various Objects

Table 10.2 Rotational Inertias

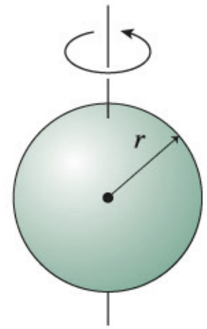


Thin rod about center
$I = \frac{1}{12}ML^2$

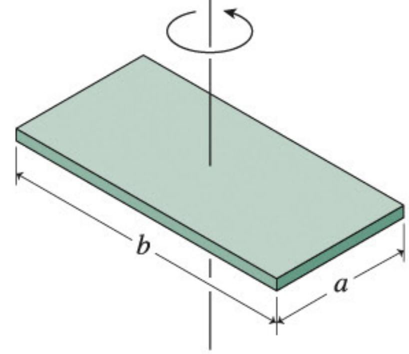


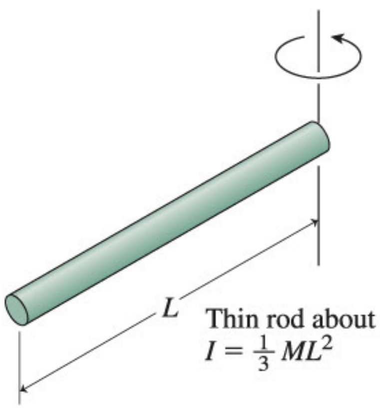
Thin ring or hollow cylinder about its axis
$I = MR^2$

Solid sphere about diameter
$I = \frac{2}{5}MR^2$

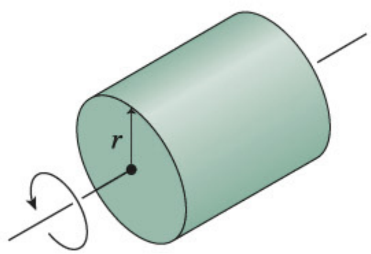


Flat plate about perpendicular axis
$I = \frac{1}{12}M(a^2 + b^2)$



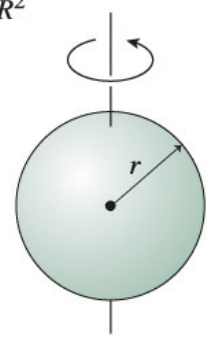


Thin rod about end
$I = \frac{1}{3}ML^2$

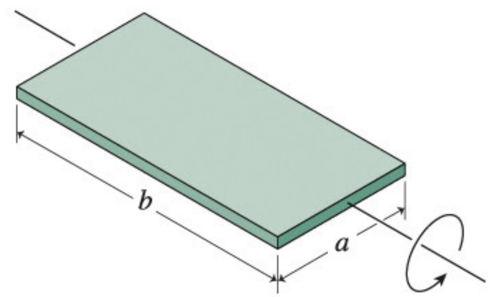


Disk or solid cylinder about its axis
$I = \frac{1}{2}MR^2$

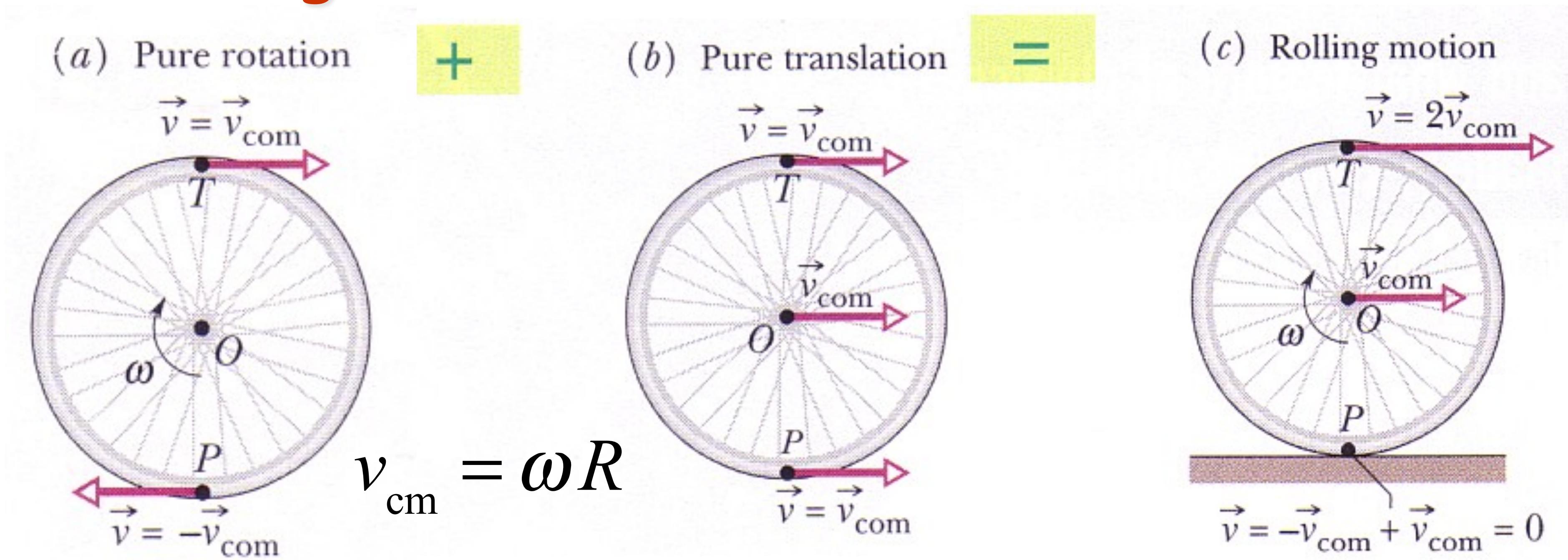
Hollow spherical shell about diameter
$I = \frac{2}{3}MR^2$



Flat plate about central axis
$I = \frac{1}{12}Ma^2$



Rolling motion as rotation and translation



$$v_{\mathrm{cm}} = \omega R$$

Kinetic energy consists of rotational & translational terms:

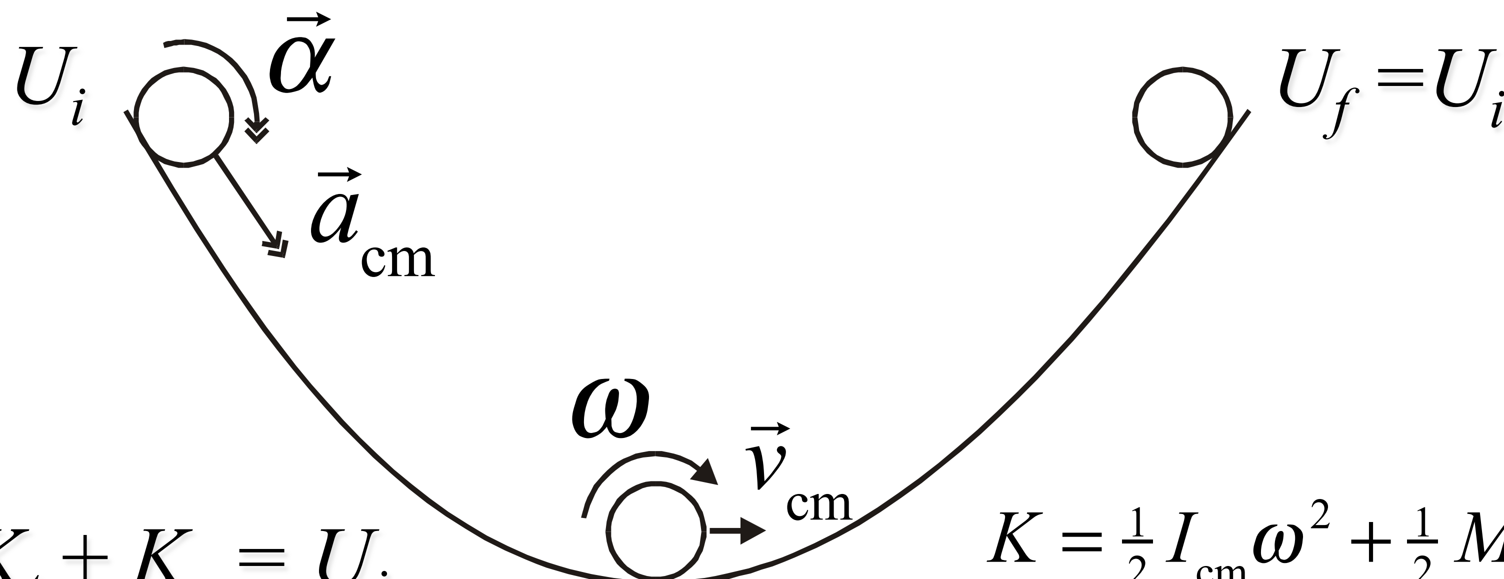
$$K = \tfrac{1}{2} I_{\mathrm{cm}} \omega^2 + \tfrac{1}{2} M v_{\mathrm{cm}}^2 = K_r + K_t$$

$$K = \tfrac{1}{2}\left\{ f M R^2 \right\} \frac{v_{\mathrm{cm}}^2}{R^2} + \tfrac{1}{2} M v_{\mathrm{cm}}^2 = \tfrac{1}{2} M' v_{\mathrm{cm}}^2$$

Modified mass: $M' = \left(1 + f\right) M$ **(look up *f* in Table 10.2)**

Rolling Motion, Friction, & Conservation of Energy

- Friction plays a crucial role in rolling motion:
    - without friction a ball would simply slide without rotating;
    - Thus, friction is a necessary ingredient.
- However, if an object rolls without slipping, mechanical energy is <u>NOT</u> lost as a result of frictional forces, which do <u>NO</u> work.
    - An object must slide/skid for the friction to do work.
- Thus, if a ball rolls down a slope, the potential energy is converted to translational and rotational kinetic energy.



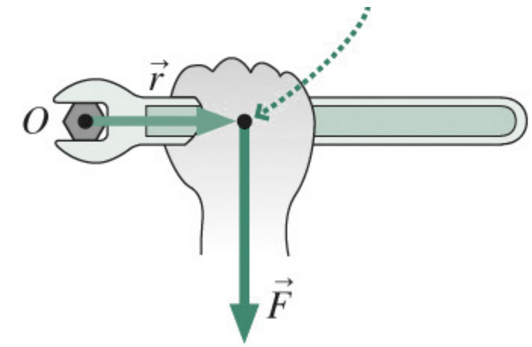
$$K = K_t + K_r = U_i$$

$$K = \tfrac{1}{2} I_{cm}\omega^2 + \tfrac{1}{2} M v_{cm}^2$$

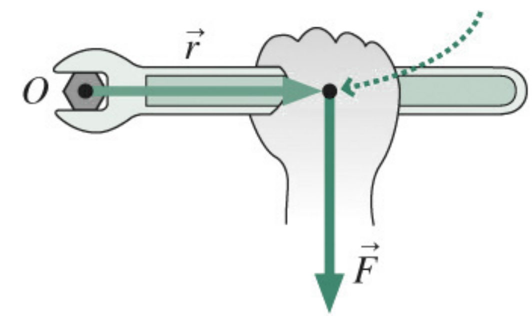
Torque and Newton's 2nd Law

- Torque (τ) is the rotational analog of force, and results from the application of one or more forces.
- Torque depends on the rotation axis.
- Torque also depends on:
  - the magnitude of the force;
  - the distance from the rotation axis to the force application point;

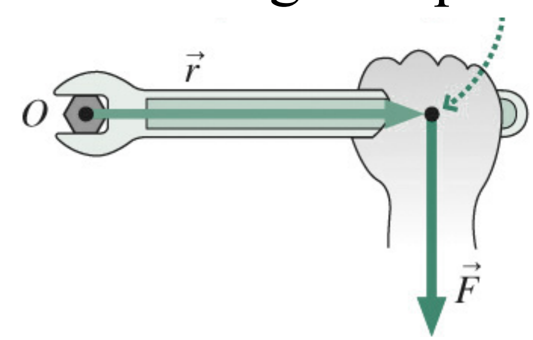
Small torque



Medium torque

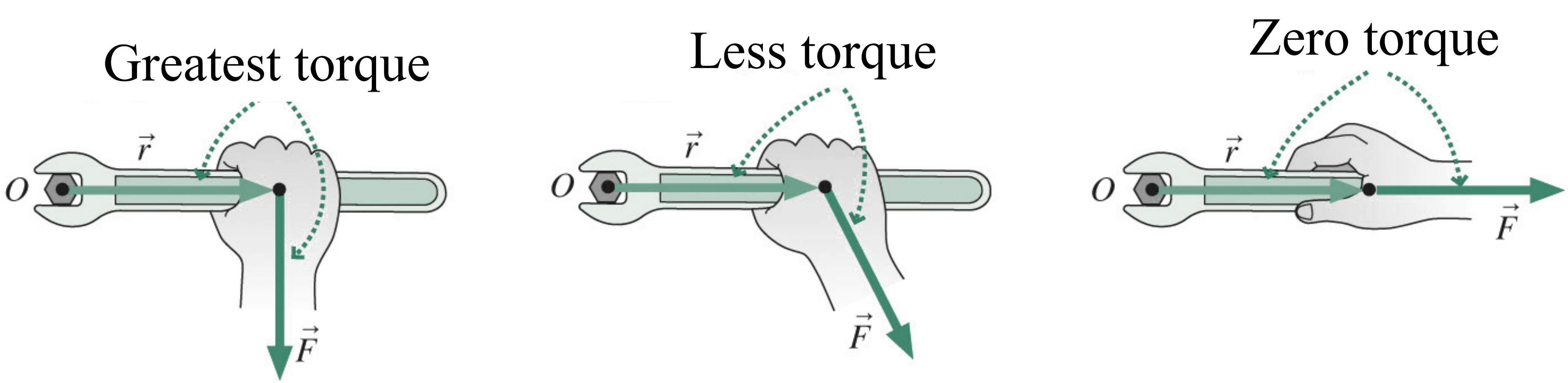


Large torque

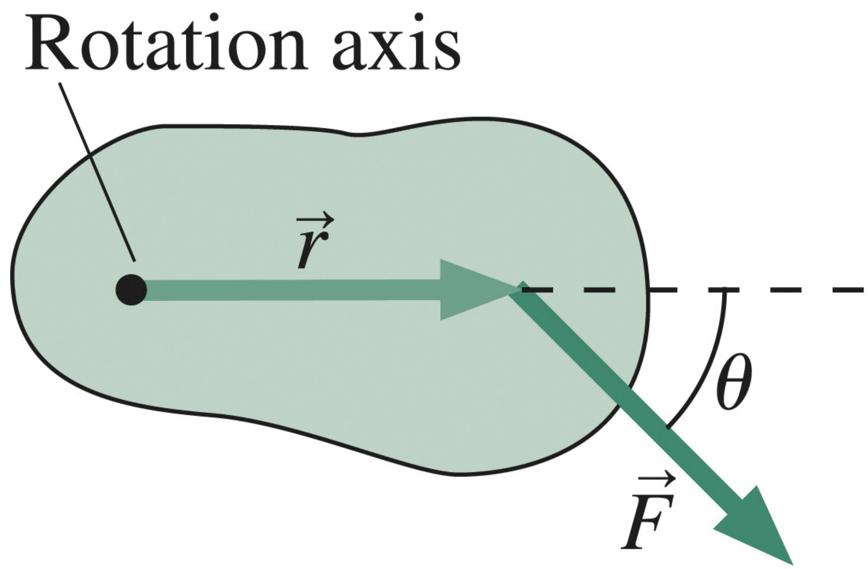


Torque and Newton's 2nd Law

- Torque (τ) is the rotational analog of force, and results from the application of one or more forces.
- Torque depends on the rotation axis.
- Torque also depends on:
  - and the orientation of the force relative to the displacement from the axis to the force application point.



Torque and Newton's 2nd Law

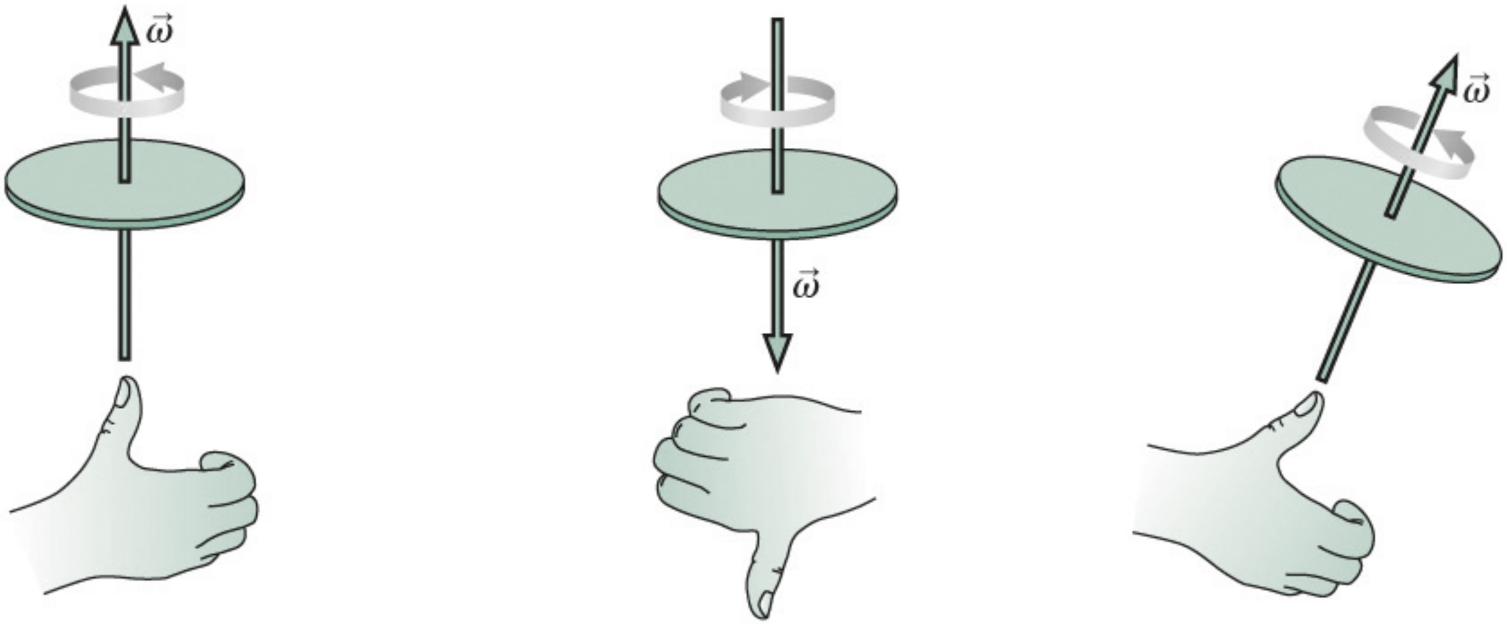


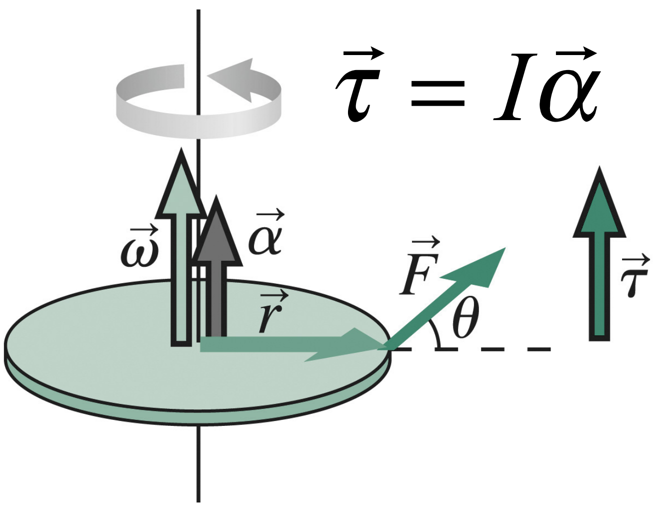
Definition: $$\tau = \left|\vec{r} \times \vec{F}\right| = rF\sin\theta$$

Newton's 2nd law: $$\tau = I\alpha$$

Angular quantities have direction

The direction of angular velocity is given by the **right-hand rule.**





$$\vec{\tau} = I\vec{\alpha}$$

Same applies to torque:

Torque is perpendicular to both the force vector and the displacement vector from the rotation axis to the force application point.

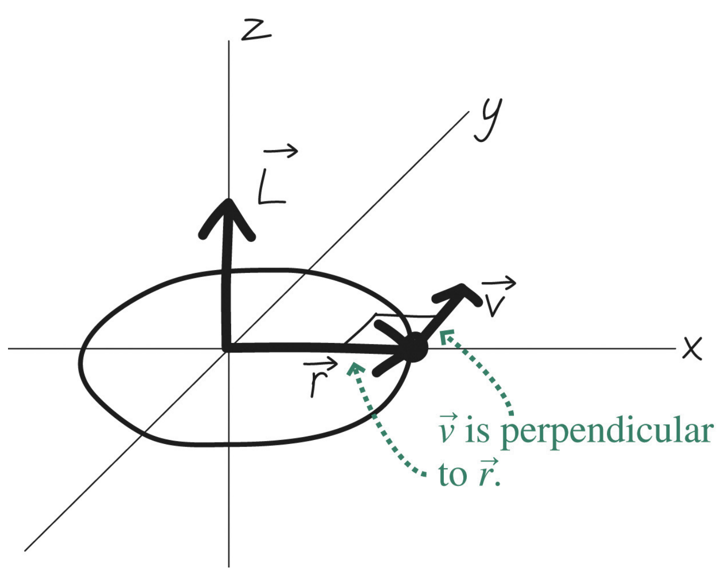
$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \left(\left|\vec{\tau}\right| = rF\sin\theta\right)$$

Angular Momentum

- For a single particle, angular momentum is a vector given by the cross product of the displacement vector from the rotation axis with the linear momentum of the particle:

Angular momentum $\vec{L}$ is defined as: $\vec{L} = \vec{r} \times \vec{p} = m\left(\vec{r} \times \vec{v}\right)$

$$L = rp\sin\phi = mvr\sin\phi$$



– For the case of a particle in a circular path, $L = mvr$, and is upward, perpendicular to the circle.

– For sufficiently symmetric objects, angular momentum is the product of rotational inertia (a scalar) and angular velocity (a vector):

$$\vec{L} = I\vec{\omega}$$

- SI unit is $Kg.m^2/s$.

Conservation of angular momentum

It follows from Newton's second law that:

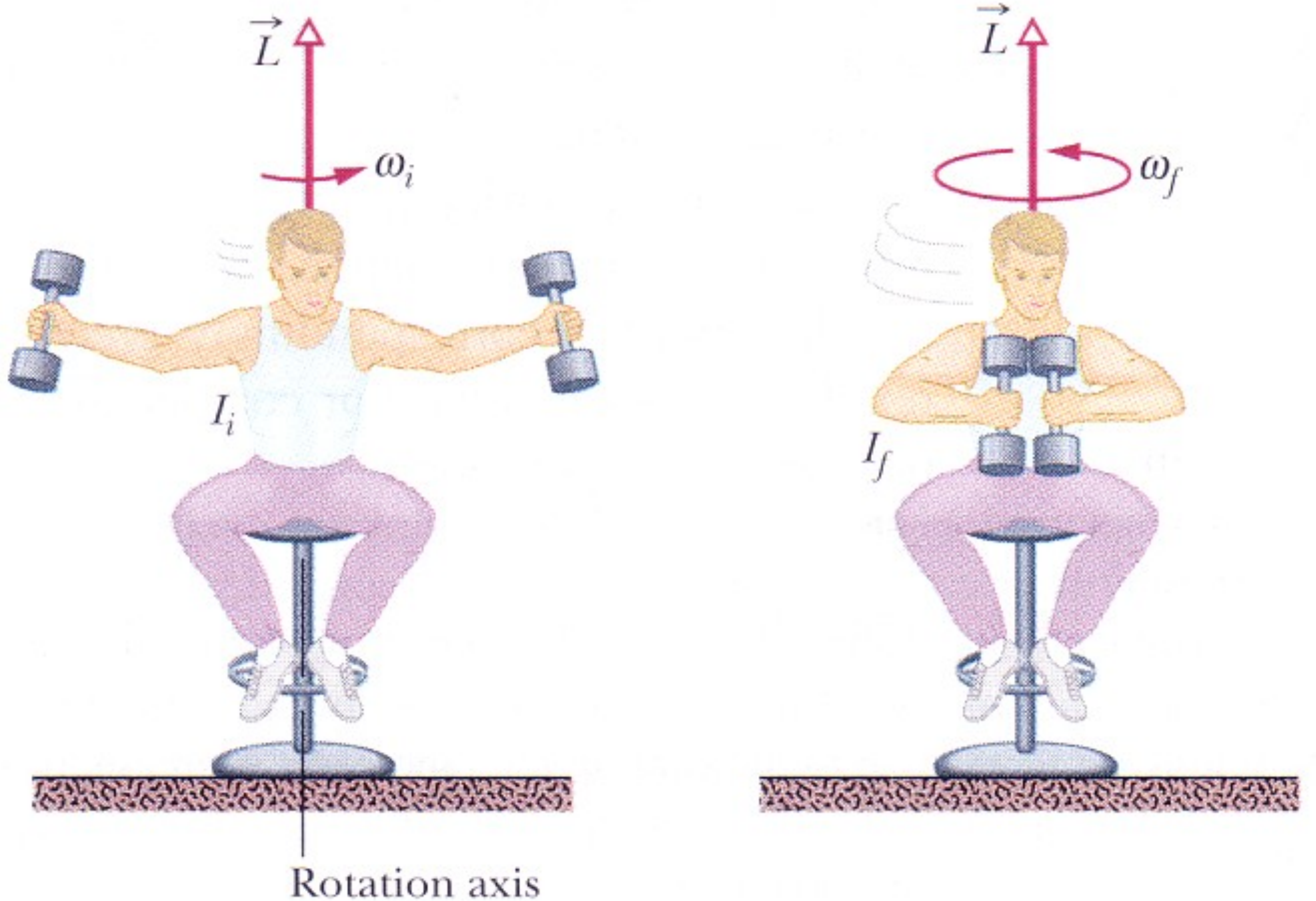
If the net external torque acting on a system is zero, the angular momentum of the system remains constant, no matter what changes take place within the system.

$$\vec{L} = \text{a constant}$$

$$\vec{L}_i = \vec{L}_f$$

$$I_i\omega_i = I_f\omega_f$$

$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$



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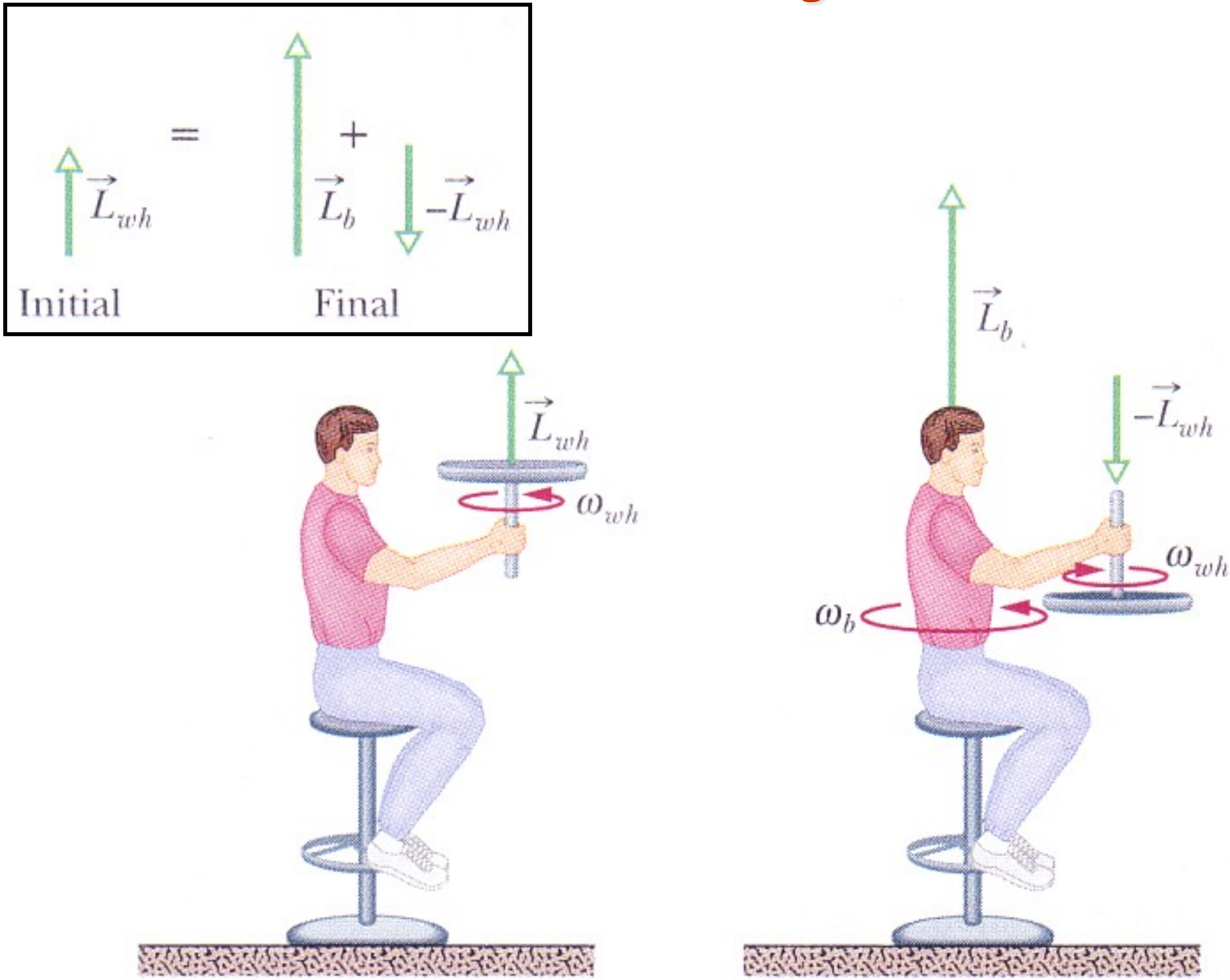
$$\frac{\omega_f}{\omega_i} = \frac{I_i}{I_f}$$

What happens to kinetic energy?

$$K_f = \tfrac{1}{2}I_f\omega_f^2 = \tfrac{1}{2}I_f\left(\frac{I_i^2\omega_i^2}{I_f^2}\right) = \frac{I_i}{I_f}\tfrac{1}{2}I_i\omega_i^2 = \frac{I_i}{I_f}K_i$$

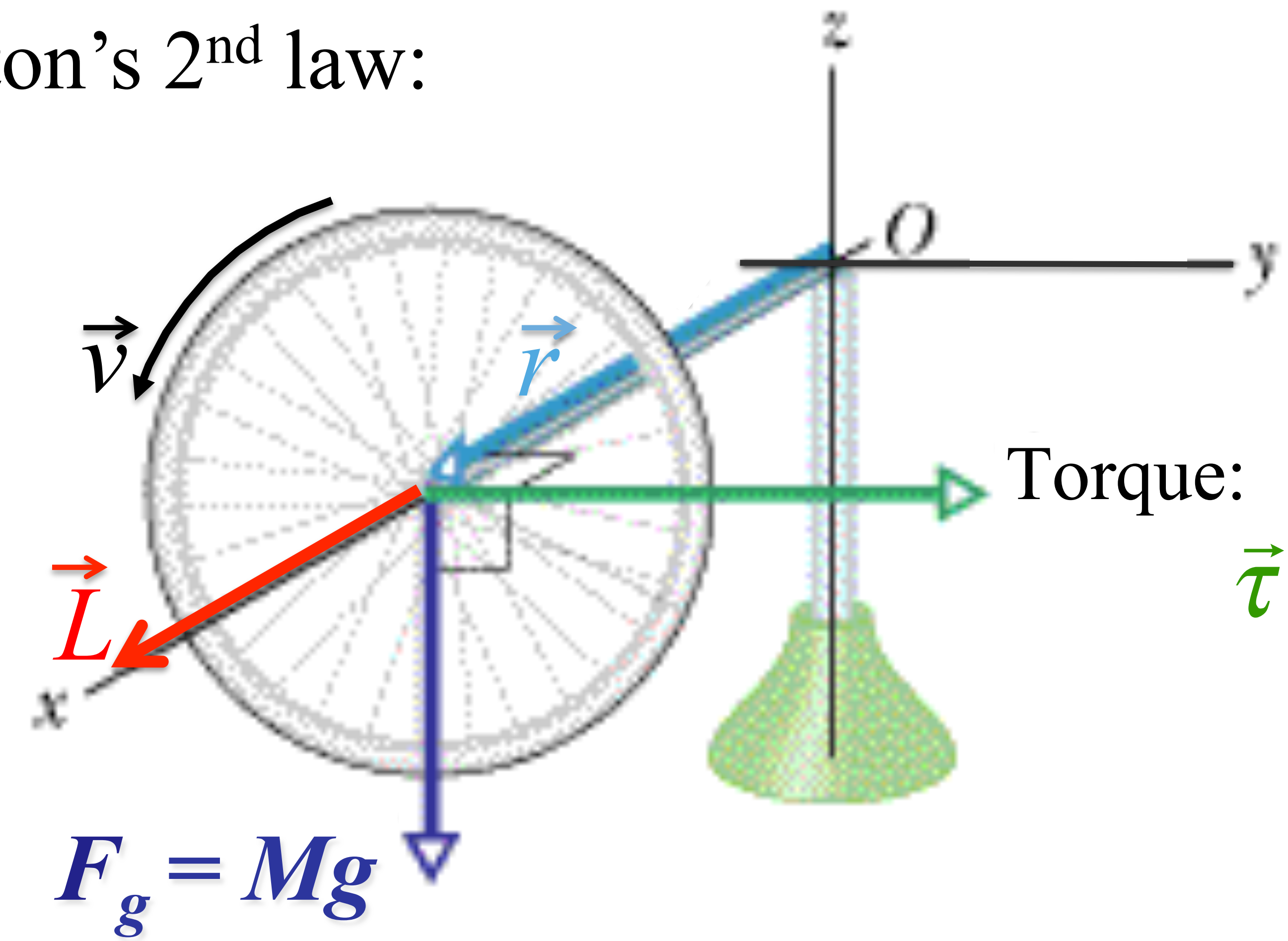
- Thus, if you increase ω by reducing I, you end up increasing K
- Therefore, you must be doing some work
- This is a very unusual form of work that you do when you move mass radially in a rotating frame
- The frame is accelerating, so Newton's laws do not hold in this frame

More on conservation of angular momentum



The Gyroscope

Newton's 2nd law:



Torque:

$$\vec{\tau} = \vec{r} \times F$$

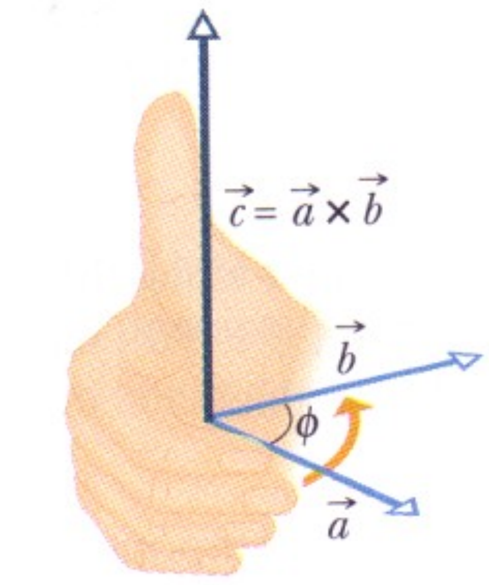
$$= \frac{d\vec{L}}{dt}$$

Used in navigational devices - even modern ones.

Summarizing relations for translational and rotational motion

Pure Translation (Fixed Direction)		Pure Rotation (Fixed Axis)	
Position	x	Angular position	θ
Velocity	$v = dx/dy$	Angular velocity	$\omega = d\theta/dt$
Acceleration	$a = dv/dt$	Angular acceleration	$\alpha = d\omega/dt$
Mass	m	Rotational inertia	I
Newton's second law	$F_{net} = ma$	Newton's second law	$\tau_{net} = I\alpha$
Work	$W = \int F\, dx$	Work	$W = \int \tau\, d\theta$
Kinetic energy	$K = \frac{1}{2}mv^2$	Kinetic energy	$K = \frac{1}{2}I\omega^2$
Power (constant force)	$P = Fv$	Power (constant torque)	$P = \tau\omega$
Work–kinetic energy theorem	$W = \Delta K$	Work–kinetic energy theorem	$W = \Delta K$

The vector product, or cross product



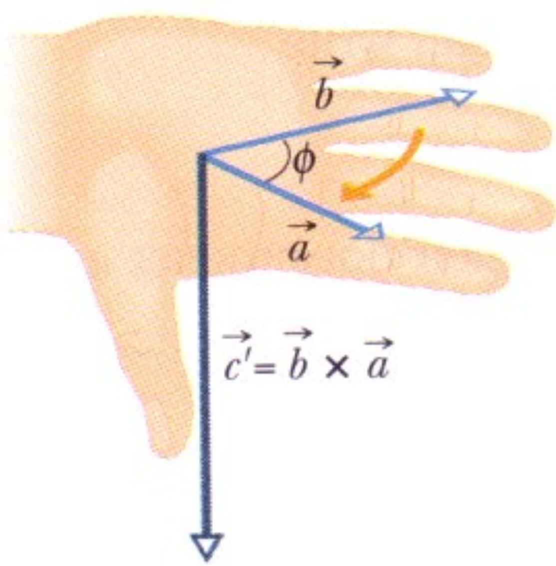
(a)

$$\vec{a} \times \vec{b} = \vec{c}, \text{ where } c = ab\sin\phi$$

$$\vec{a} \times \vec{b} = -\left(\vec{b} \times \vec{a}\right)$$

Direction of $\vec{c} \perp$ to both $\vec{a}$ and $\vec{b}$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$



$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$

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$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j}$$